

FIAN/TD -21/95

December 1995

**NONPERTURBATIVE DISSIPATION IN QCD JETS****A.V. Leonidov <sup>1</sup> and D.M. Ostrovsky <sup>2</sup>**

*Theoretical Physics Department, P.N.Lebedev Physics Institute 117924 Leninsky pr. 53, Moscow,  
Russia*

**Abstract**

The phenomenological energy dissipation in QCD jet evolution taking into account the colour coherence is considered and the corresponding modified parton multiplicity and distribution function are computed.

---

<sup>1</sup>e-mail address leonidov@td.lpi.ac.ru

<sup>2</sup>e-mail address ostrov@td.lpi.ac.ru

# 1 Introduction

Physics of QCD jets is one of the most important testing grounds for the theory of strong interactions. The perturbative evolution of a quark-gluon jet is at present well understood (see, e.g., [1]). The known details include in particular the famous colour coherence phenomenon which leads to a probabilistic picture of timelike jet evolution where the daughter partons are emitted into a gradually shrinking cones. The nonperturbative aspects of a jet evolution are much less understood. From the experience of a sum rule approach applied to the analysis of heavy resonance properties [2] it is clear, that when the virtuality of a daughter parton is of order of 1 GeV one has to take into account the nonperturbative corrections due to the nonperturbative vacuum fields giving rise to the QCD vacuum condensates. Another situation in which one deals with dissipative effects is a jet propagation in, e.g., nuclear medium. Although physically these situations are very different, we expect the corresponding theoretical formalism to be the same.

In both cases the new interaction vertices lead to the appearance of new dimensionful parameters. This means in turn that a scaling description of jet evolution where the interaction vertices are determined only by dimensionless quantities (energy or virtuality ratios) is no longer valid. Physically what happens is a beginning of a string formation taking energy from the perturbative component and converting it to nonperturbative degrees of freedom. The first model description of a jet evolution taking into account the nonperturbative energy loss was proposed long ago by Dremin [3]. It was based on the analogy with the physics of high energy electromagnetic showers in the medium, where apart from a scale-invariant evolution due to a photon bremsstrahlung and pair creation there appears a scale-noninvariant energy dissipation due to the ionization of the atoms in the medium [4]. Later the corresponding modified evolution equations were analytically solved in [5], where the expressions for the parton multiplicities in quark and gluon jets and the energy loss by the perturbative component were computed. A discussion of this approach and also of the latest Monte-Carlo calculations, [6] where the nonperturbative component was described by QCD effective lagrangeans can be found in the recent review [7]. A serious drawback of Dremin equations [3] is, however, a disregard of the kinematical and color interference effects in the jet evolution which are known to be of crucial importance for describing the jet characteristics. The main goal of the present paper is to introduce a formalism providing a possibility of an analytical treatment of the jet evolution taking into account both color interference and nonperturbative energy loss.

Before writing down the modified evolution equations let us recall, that color coherence in QCD jets can be described using various approximations. The simplest is a leading logarithmic one (DLA, see [1]). The calculations in this approximation are relatively simple, but the energy of the jet is not conserved. This is due to the eikonal description used in this approximation, when the energy loss of the projectile is considered to be negligible. A more refined analysis is made within a modified leading logarithmic approximation (MLLA), where the total energy is conserved. The

Dremin equations [3] correspond to a case, where the energy is perturbatively conserved. Thus to calculate an absolute value of the dissipative energy loss one has to modify an MLLA formalism. The corresponding equations can be written (see Appendix A), but unfortunately we were not able to find their solutions. Thus in what follows we shall concentrate on a simpler DLA case, where the energy nonconservation is built in already at the level of perturbation theory. In this case one can not calculate the absolute energy loss, but the relative one is fairly well-defined.

## 2 DLA Equation for the Generating Functional

Let us turn to a description of a formalism which generalizes the known effective methods of describing the QCD jet evolution including color coherence effect [1] by providing a possibility of considering the new possible nonperturbative sources of energy dissipation.

Technically it is most convenient to use a generating functional.  $G(\{u\})$  from which one can calculate both an exclusive cross section

$$d\sigma_N^{excl} = \left( \prod_{i=1}^N d^3k_i \frac{\delta}{\delta u(k_i)} \right) G(\{u\}) \Big|_{u=0} d\sigma_0. \quad (1)$$

and an inclusive one

$$d\sigma_N^{incl} = \left( \prod_{i=1}^N d^3k_i \frac{\delta}{\delta u(k_i)} \right) G(\{u\}) \Big|_{u=1} d\sigma_0. \quad (2)$$

In the following we shall use the notations  $dK = \gamma_0^2 \frac{dk}{k} \frac{d^2k_\perp}{2\pi k_\perp^2}$  for the usual DLA integration measure and  $\Gamma(p, \theta)$  for the possible gluon emission phase space domain, where  $p$  is an initial gluon energy and  $\theta$  is an opening angle of a jet. Let us remind, that in DLA approximation the generating functional  $G(p, \theta; \{u\})$  satisfies the following master equation

$$G(p, \theta; \{u\}) = u(p) e^{-w(p, \theta)} + \int_{\Gamma(p, \theta)} \frac{dk}{k} \frac{d^2k_\perp}{k_\perp^2} \frac{2C_V \alpha_s}{\pi} e^{-w(p, \theta) + w(p, \theta_k)} G(p, \theta_k; \{u\}) G(k, \theta_k; \{u\}), \quad (3)$$

where

$$w(p, \theta) = \int_{\Gamma(p, \theta)} dK$$

is a total probability of an emission of a gluon having an energy  $p$  into the cone having an opening angle  $\theta$ , and  $\exp(-w(p, \theta))$  is a formfactor giving a probability of a non-emission of the same gluon. Then a standard procedure ([1]) leads to the following equation on the generating functional:

$$G(p, \theta; \{u\}) = u(p) \exp \left( \int_{\Gamma(p, \theta)} dK [G(k, \theta_k; \{u\}) - 1] \right). \quad (4)$$

Let us now discuss the possible nonperturbative modifications of the equations for the generating functional. The aim is to describe a partial conversion of the perturbative degrees of freedom

(gluons) and their energy into the nonperturbative ones by the process presumably analogous to the ionization in the case of electromagnetic showers discussed in the Introduction. Physically the picture for the jet is that the dissipation pumps some energy from the cone corresponding to the perturbative jet evolution reducing the gluon multiplicity and energy inside it. Thus we have some additional nonperturbative vertex coupling the perturbative gluons with the nonperturbative vacuum or nuclear medium. The presence of such a coupling can be related to reducing a probability of an emission of a perturbative gluon by taking away a certain portion of its energy before the perturbative emission. Technically it is convenient to introduce a following modification of the gluon emission probability:

$$w(p, \theta; \beta) = \int_{\Gamma(p-\beta, \theta)} dK, \quad (5)$$

where the constant  $\beta$  describes the additional damping of a perturbative gluon emission due to a dissipative interaction with the nonperturbative vacuum fluctuations or nuclear medium introduced by compressing a possible phase space domain for the gluon emission from  $\Gamma(p, \theta)$  to  $\Gamma(p - \beta, \theta)$ . The corresponding modification of the master equation (3) reads

$$G(p, \theta; \{u\}) = u(p)e^{-w(p, \theta; \beta)} + \int_{\Gamma(p-\beta, \theta)} \frac{dk}{k} \frac{d^2 k_{\perp}}{k_{\perp}^2} \frac{2C_V \alpha_s}{\pi} e^{-w(p, \theta; \beta) + w(p, \theta_k; \beta)} G(p, \theta_k; \{u\}) G(k, \theta_k; \{u\}), \quad (6)$$

Then instead of the Eq. (4) we get

$$G(p, \theta; \{u\}) = u(p) \exp \left( \int_{\Gamma(p-\beta, \theta)} dK [G(k, \theta_k; \{u\}) - 1] \right), \quad (7)$$

where the initial conditions for this equation read  $G(p, \theta; \{u\})|_{u=1} = 1$ .

We suggest that  $\beta \ll p$  for all physically meaningful values of energy and in the following we shall consider the terms of the first order in the small parameter  $\varepsilon = \beta/p$ . In Appendix B we illustrate the thus arising perturbation theory by deriving the differential equation on the leading correction to the generating functional.

### 3 Mean multiplicity

Let us begin this section by looking at the simplest jet characteristic, a mean multiplicity of partons in a jet (in the following we are considering only gluons). Following the standard steps we obtain a following integral equation on the mean multiplicity  $\bar{n}(p, \theta)$ :

$$\bar{n}(p, \theta) = 1 + \int_{\Gamma(p-\beta, \theta)} dK \bar{n}(k, \theta_k). \quad (8)$$

In the following it will be convenient to rewrite the integration over the phase space as

$$\int_{\Gamma(p-\beta, \theta)} dK = \int_{\Gamma(p-\beta, \theta)} \frac{dk}{k} \frac{d^2 k_{\perp}}{2\pi k_{\perp}^2} \gamma_0^2 = \int_0^{y-\varepsilon} d\xi \int_0^{\xi} dy' \gamma_0'^2, \quad (9)$$

where

$$y = \ln \frac{p\theta}{Q_0}, y' = \ln \frac{k\theta_k}{Q_0}, \xi = \ln \frac{k\theta}{Q_0}.$$

Working in the first order in  $\varepsilon$  we can write

$$\bar{n}(p, \theta) = \bar{n}_0(y) + \varepsilon \bar{n}_1(y).$$

The equations for the functions  $\bar{n}_0(y)$  and  $\bar{n}_1(y)$  take the form

$$\bar{n}_0(y) = 1 + \int_0^y dy' \gamma_0^2 \bar{n}_0(y') \quad (10)$$

and

$$\bar{n}_1(y) = \int_0^y dy' \gamma_0^2 \bar{n}_1(y') (e^{y-y'} - 1) - \int_0^y dy' \gamma_0^2 \bar{n}_0(y') \quad (11)$$

The first one has a well known solution

$$\bar{n}_0(y) = \text{ch}(\gamma_0 y),$$

and the second one can be rewritten in the differential form

$$\bar{n}_1''(y) - \bar{n}_1'(y) = \gamma_0^2 \bar{n}_1(y) + \gamma_0^2 (\text{ch}(\gamma_0 y) - \gamma_0 \text{sh}(\gamma_0 y)) \quad (12)$$

with the initial conditions  $\bar{n}(0) = 0, \bar{n}'(0) = -\gamma_0^2$ . The solution reads

$$\bar{n}_1 = \gamma_0^2 \text{ch}(\gamma_0 y) - \gamma_0 \text{sh}(\gamma_0 y) + \gamma_0 \frac{\lambda_2 e^{\lambda_1 y} - \lambda_1 e^{\lambda_2 y}}{\sqrt{1 + 4\gamma_0^2}}. \quad (13)$$

In the limit of ( $y \gg 1, \gamma_0 \ll 1$ ) we have

$$\bar{n} \approx \frac{1}{2} e^{\gamma_0 y} (1 - \varepsilon \gamma_0 - 2\varepsilon \gamma_0^3 e^y) = \frac{1}{2} \left( \frac{Q}{Q_0} \right)^{\gamma_0} \left( 1 - \gamma_0 \frac{\beta}{p} - 2\gamma_0^3 \frac{\beta \theta}{Q_0} \right) \quad (14)$$

The above result is in accord with the expectation based on physical reasoning. Namely, the dissipation should lead to a decrease in perturbative multiplicity. Let us note, that from Eq. (14) it is clear, that in order for perturbation theory to be applicable one should have  $\beta \ll p/\gamma_0$  and  $\beta \ll Q_0/\gamma_0^3$ . This means that apart from the expected expansion parameter  $\varepsilon$  there appears a new one  $\varepsilon' = 2\gamma_0^2 \frac{\beta \theta}{Q_0}$ . An interesting feature of the above answer is a multiplicative combination of the dissipation scale  $\beta$  and a perturbative coupling constant hidden in  $\gamma_0$ .

## 4 Energy distrubution of particles in a jet

Let us now turn to a computation of a one-particle energy distribution function  $\bar{D}(k, \theta)$ . It can be obtained from the generating functional in the following way:

$$\bar{D}(k, \theta) = k \frac{\delta}{\delta u(k)} G(p, \theta; \{u\}) \Big|_{u=1}, \quad (15)$$

Making use of (7) and (3) we get

$$\bar{D}(l, y) = \delta(l) + \int_0^{l-\varepsilon} dl' \int_0^y dy' \gamma_0^2 \bar{D}(l', y'), \quad (16)$$

where  $l = \ln(p/k) = \ln(1/x)$  and  $l' = \ln(k'/k)$ . The integration is performed over  $k'$  and  $\theta_{k'}$ . Let us stress that in the above equation (16) the distrubution function  $\bar{D}$  is also  $\beta$  - dependent. Expanding it in the small parameter  $\varepsilon$

$$\bar{D}(l, y) = \bar{D}_0(l, y) + \varepsilon \bar{D}_1(l, y)$$

we rewrite (16) in the form

$$\bar{D}_0(l, y) + \varepsilon \bar{D}_1(l, y) = \delta(l) + \int_0^{l-\varepsilon} dl' \int_0^y dy' \gamma_0^2 (\bar{D}_0(l', y) + \varepsilon e^{l-l'} \bar{D}_1(l', y)). \quad (17)$$

Considering the zero and first order in  $\varepsilon$  we have

$$\bar{D}_0(l, y) = \delta(l) + \int_0^l dl' \int_0^y dy' \gamma_0^2 \bar{D}_0(l', y'). \quad (18)$$

$$\bar{D}_1(l, y) = \int_0^l dl' \int_0^y dy' \gamma_0^2 e^{l-l'} \bar{D}_1(l', y') - \int_0^y dy' \gamma_0^2 \bar{D}_0(l, y'). \quad (19)$$

The solution of equation (18) reads

$$\bar{D}_0(l, y) = \delta(l) + \gamma_0 \sqrt{\frac{y}{l}} I_1(2\gamma_0 \sqrt{yl}) \quad (20)$$

and that for  $\bar{D}_1$  is (for details see Appendix C):

$$\bar{D}_1(l, y) = -y\gamma_0^2 \delta(l) - \gamma_0^2 (e^l + 1) \frac{y}{l} I_2(2\gamma_0 \sqrt{yl}) + \gamma_0^3 \left(\frac{y}{l}\right)^{3/2} (e^l - 1) I_3(2\gamma_0 \sqrt{yl}). \quad (21)$$

The plot of the resulting distribution for the jet energy 20 GeV is given at Fig. 1. We see that, as anticipated, the effect of dissipation shows itself through the noticeable reduction of distribution function. Let us also mention that here we also have a multiplicative dependence on both perturbative and nonperturbative factors.

## 5 Conclusions

In this paper we have proposed a phenomenological nonperturbative modification of the equations describing the evolution of QCD jets and taking into account the crucial feature of the colour coherence of the QCD cascades by accounting for nonperturbative energy dissipation. The corresponding modification of the MLLA and DLA formalism exploiting the generating functional was proposed. The calculation of simplest jet characteristics such as mean multiplicity and energy spectrum of particles in a jet in a DLA approximation has demonstrated an expected decrease in multiplicity and corresponding changes in the energy distribution.

An interesting feature of the result is an unusual perturbative damping of the introduced non-perturbative energy dissipation appearing in the multiplicative dependence on some power of the QCD coupling constant times the dissipative scale. It is tempting to relate this feature with the successes of the soft blanching hypothesis, where it is assumed that the nonperturbative effects are not crucially essential for the jet characteristics. It is interesting to see, whether such damping is present within a more realistic MLLA description. Work in this direction is in progress.

## 6 Acknowledgements

We are grateful to I.M. Dremin and I.V. Andreev for the useful discussions. A.L. is grateful to P.V. Ruuskanen for kind hospitality in the University of Helsinki, where part of this work was done. The research was supported by Russian Fund for Fundamental Research, Grant 93-02-3815.

## References

- [1] Yu.L. Dokshitzer, V.A. Khose, A.H. Mueller and S.I. Troyan *Basics of Perturbative QCD* (France: Gif-sur-Yvette, Edition Frontiers, 1991)
- [2] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys.* **B147** (1979), 385
- [3] I.M. Dremin, *Pisma v ZHETP* **31** (1980), 201
- [4] S.Z. Belenky, *Shower Processes in Cosmic Rays*, M., Gostehizdat, 1948
- [5] I.M. Dremin, A.V. Leonidov *Yad. Fiz.* **35** (1980), 430
- [6] J. Ellis, K. Geiger, *Preprint CERN* **CERN-TH 95-34, 95-35** (1995)
- [7] I.M. Dremin, A.V. Leonidov *Usp. Fiz. Nauk* **7** (1995), 759

## Appendix A

Let us start with the proposed general modification of the equation for the generating functional (we are closely following the standard notations of [1]):

$$\begin{aligned}
G_A(p, \theta; \{u\}) &= u(p) e^{-w_A(p, \theta)} + \int_{\Gamma(p, \theta)} dK \gamma_0^2(k_\perp) e^{-w_A(p, \theta) + w_A(p, \theta_k)} * \\
&* \left[ \frac{1}{2} K_A^{BC} \left( \frac{k}{p} \right) G_B(k, \theta_k; \{u\}) G_C(p - k, \theta_k; \{u\}) - \right. \\
&\quad \left. - L_A^B(k, p) G_B(k, \theta_k; \{u\}) \right].
\end{aligned} \tag{A.1}$$

Here  $L_A^B(k, p)$  is a phenomenologically introduced vertex responsible for the nonperturbative energy dissipation. It is convenient to rewrite the above equation in the form:

$$\begin{aligned}
G_A(p, \theta; \{u\}) e^{w_A(p, \theta)} &= u(p) + \int_{\Gamma(p, \theta)} dK \gamma_0^2(k_\perp) e^{w_A(p, \theta_k)} * \\
&* \left[ \frac{1}{2} K_A^{BC} \left( \frac{k}{p} \right) G_B(k, \theta_k; \{u\}) G_C(p - k, \theta_k; \{u\}) - \right. \\
&\quad \left. - L_A^B(k, p) G_B(k, \theta_k; \{u\}) \right].
\end{aligned} \tag{A.2}$$

This equation has to be supplied by the boundary condition  $G_A = 1$  at  $u = 1$ . This gives an equation of the formfactor  $w_A(p, \theta)$ :

$$e^{w_A(p, \theta)} = 1 + \int_{\Gamma(p, \theta)} dK \gamma_0^2(k_\perp) e^{w_A(p, \theta_k)} \left[ \frac{1}{2} \sum_{B, C} K_A^{BC} \left( \frac{k}{p} \right) - \sum_B L_A^B(k, p) \right],$$

where  $z = \frac{k}{p}$ . Standard procedure leads us to the following master equation:

$$\begin{aligned}
\frac{\partial G_A(p, \theta)}{\partial \ln \theta} &= \int_0^1 dz \gamma_0^2(k_\perp) \left[ \frac{1}{2} K_A^{BC}(z) \left\{ G_B(zp, \theta) G_C((1-z)p, \theta) - \sum_{B, C} G_A(p, \theta) \right\} - \right. \\
&\quad \left. - L_A^B(z \cdot p, p) \left\{ G_B(zp, \theta) - \sum_B G_A(p, \theta) \right\} \right]
\end{aligned} \tag{A.3}$$

Let us now consider a pure quodynamics and a specific form of the nonperturbative vertex  $L_A^B(z \cdot p, p) = \nu \delta(1 - \frac{\beta'}{p} - z)$ . Then the master equation takes the form

$$\begin{aligned}
\frac{\partial G}{\partial \ln \theta} &= \int_0^1 dz \gamma_0^2(k_\perp) \left[ \frac{1}{2} K(z) \{ G(zp, \theta) G((1-z)p, \theta) - G(p, \theta) \} - \right. \\
&\quad \left. - \nu \gamma_0^2(k_\perp) \{ G(p - \beta', \theta) - G(p, \theta) \} \right]
\end{aligned}$$



In the interesting case of  $\beta \ll p$  we get

$$G(p - \beta', \theta) - G(p, \theta) \approx -\beta' \frac{\partial G}{\partial p}$$

and

$$\frac{\partial G}{\partial \ln \theta} = \int_0^1 dz \gamma_0^2(k_\perp) \left[ \frac{1}{2} K(z) \{G(zp, \theta) - G((1-z)p, \theta) - G(p, \theta)\} \right] + \beta' \nu \gamma_0^2 \frac{\partial G}{\partial p}. \quad (\text{A.4})$$

The above equation contains a characteristic derivative over energy (the last term). This term is responsible for the energy dissipation and has previously been used for describing the dissipative energy losses in the electromagnetic showers [4] and QCD cascades [3], [5].

## Appendix B

In this Appendix we derive an equation for the first correction to the generating functional. Let us start with the basic equation

$$G(p, \theta; \{u\}) = u(p) \exp \left( \int_{\Gamma(p-\beta, \theta)} dK [G(k, \theta_k; \{u\}) - 1] \right) \quad (\text{B.1})$$

with the initial condition  $G(p, \theta; \{u\})|_{u=1} = 1$ . Expanding the generating functional we get in the zeroth and first orders in  $\varepsilon$ :

$$G(p, \theta; \{u\}) = G_0(y; u) + \varepsilon G_1(y; u), \quad (\text{B.2})$$

where  $y = \ln(p\theta/Q_0)$ ,  $G_0(y; 1) = 1$ ,  $G_1(y; 1) = 0$  and  $\varepsilon = \beta/p$ .

An argument of the exponent in (B.1) is

$$\begin{aligned} & \int_0^y dy' \gamma_0^2(y - y') (G_0(y'; u) - 1) - \varepsilon \int_0^y dy' \gamma_0^2(y - y') (G_0(y'; u - 1)) - \\ & - \varepsilon \int_0^y dy' \gamma_0^2(y - y') G_1(y'; u). \end{aligned} \quad (\text{B.3})$$

After expanding this exponent to the first order in  $\varepsilon$  we arrive at

$$G_0(y; u) = u \exp \left[ \int_0^y dy' \gamma_0^2(y - y') (G_0(y'; u) - 1) \right] \quad (\text{B.4})$$

and

$$G_1(y; u) = G_0(y; u) \left[ \int_0^y dy' \gamma_0^2 (e^{y-y'} - 1) G_1(y'; u) - \int_0^y dy' \gamma_0^2 (G_0(y'; u) - 1) \right], \quad (\text{B.5})$$

or in a simpler form

$$G_0'(y) = G_0(y) \int_0^y dy' \gamma_0^2 (G_0(y') - 1) \quad (\text{B.6})$$

and

$$G_1(y) = G_0(y) \int_0^y dy' \gamma_0^2 (e^{y-y'} - 1) G_1(y') - G_0'(y) \quad (\text{B.7})$$

The further simplification can be achieved by introducing a new function  $H(y)$  by

$$H(y) = \int_0^y dy' (e^{y-y'} - 1) G_1(y') \quad G_1'(y) = H''(y) - H'(y) \quad (\text{B.8})$$

After the simple calculation we obtain the following differential equation on the leading correction to the generating functional

$$H''(y) - H'(y) = \gamma_0^2 G_0(y) H(y) - G_0'(y) \quad (\text{B.9})$$

The solutions of the above equation could be used as a starting point for the analysis of the dissipative corrections to the scaling behaviour of the moments and, in general, to the KNO property of the leading approximation.

## Appendix C

In this Appendix we shall give some details on the computation of the correction to the distribution function  $\bar{D}(l, y)$ . We start with the equations Eqs. (18), (19):

$$\bar{D}(l, y) = \bar{D}_0(l, y) + \varepsilon \bar{D}_1(l, y)$$

$$\bar{D}_0(l, y) = \delta(l) + \int_0^l dl' \int_0^y dy' \gamma_0^2 \bar{D}_0(l', y'). \quad (\text{C.1})$$

$$\bar{D}_1(l, y) = \int_0^l dl' \int_0^y dy' \gamma_0^2 e^{l-l'} \bar{D}_1(l', y') - \int_0^y dy' \gamma_0^2 \bar{D}_0(l, y'). \quad (\text{C.2})$$

The solution of equation (18) is performed by substituting

$$\bar{D}_0(l, y) = \delta(l) + \sum C_{m,n}^{(0)} l^m y^n,$$

giving

$$C_{m,n}^{(0)} = \delta_{m,m+1} \frac{\gamma_0^{2(m+1)}}{m!(m+1)!}$$

and

$$\bar{D}_0(l, y) = \delta(l) + \gamma_0 \sqrt{\frac{y}{l}} I_1(2\gamma_0 \sqrt{yl}) \quad (\text{C.3})$$

Let us look for the solution for  $\bar{D}_1$  in the form

$$\bar{D}_1 = -y\gamma_0^2 \delta(l) + e^l \sum C_{m,n}^{(1)} l^m y^n.$$

The solution for  $C^{(1)}$  reads

$$C_{m,n}^{(1)} = \begin{cases} -(-1)^{m+n} \frac{\gamma_0^{2n}}{n!m!} \sum_{p=0}^{n-2} C_{m-p}^{m-n+2} + \delta_{m+2,n} \left( -\frac{\gamma_0^{2(m+1)}}{m!n!} \right) & n \leq m+2 \\ 0 & \text{otherwise} \end{cases} \quad (\text{C.4})$$

Thus we obtain the correction for the energy distribution

$$\bar{D}_1(l, y) = -y\gamma_0^2 \delta(l) - e^l \left[ \frac{y}{l} I_2(2\gamma_0 \sqrt{yl}) + \sum_{m,n} (-1)^{m+n} \frac{\gamma_0^{2n}}{m!n!} \sum_{p=0}^{n-2} C_{m-p}^{m-n+2} l^m y^n \right], \quad (\text{C.5})$$

which can finally be written in a compact analytical form Eq. (21):

$$\bar{D}_1(l, y) = -y\gamma_0^2 \delta(l) - \gamma_0^2 (e^l + 1) \frac{y}{l} I_2(2\gamma_0 \sqrt{yl}) + \gamma_0^3 \left( \frac{y}{l} \right)^{3/2} (e^l - 1) I_3(2\gamma_0 \sqrt{yl}). \quad (\text{C.6})$$